

Helicity fluctuation, generation of linking number and effect on resistivity

F. Spineanu, M. Vlad
 Association EURATOM-MEC Romania
 NILPRP MG-36, Magurele, Bucharest, Romania
 spineanu@ifin.nipne.ro

February 2, 2008

Abstract

The energy of the stochastic magnetic field is bounded from below by a topological quantity expressing the degree of linkage of the field lines. When the bound is saturated one can assume that the storage of a certain magnetic energy requires a minimal degree of topological complexity. It is then possible to infer a connection between the helicity content and the average curvature of the magnetic field lines. The random curvature induce random drifts leading to an additional dissipation and modified resistivity.

1 Introduction

When the Chirikov criterion is verified for several chains of magnetic islands (developing at closely neighbor resonant magnetic surfaces in a volume of plasma) the magnetic field becomes stochastic. In general the magnetic stochasticity is taken into account in transport processes due to the high efficiency of energy spreading through the stochastic region. However from the point of view of the structure of the magnetic field it is difficult to say anything more than we know from Hamiltonian chaotic systems: there is a stochastic (exponential) instability, local Lyapunov exponents and Kolmogorov length and the test particles move diffusively or have various sub- and supra-diffusive behaviors.

However in a stochastic region the field must still obey some constraints. These constraints arise from the relation between the energy stored in the

magnetic field and the topological complexity of the field. The constraints can be briefly expressed in this way: it is not possible to support a certain energy in a volume spanned by transiently stochastic magnetic field lines if these magnetic field lines do not have a certain minimal degree of topological complexity.

This should be seen in relation with the equation that expresses the topological *link* in terms of *writhe* and *twist* and in relation with the dynamics of a twisted flux tube. If an initial amount of link is stored exclusively as twist, then beyond a certain level of twist the flux tube deforms and acquires writhe, thus distributing the higher amount of link into the two kinds of topological deformations: twist and writhe. In a plasma free from strong magnetic background (as in astrophysical plasma or solar corona) generation of writhe means a coiling or super-coiling instability, a large spatial deformation. In a tokamak the stochastic flux tubes are also subject to the writhing instability when a local fluctuation of the parallel electric field occurs, but they are more constraint by the confining \mathbf{B}_0 and cannot perform large spatial displacements. Instead, as a result of small deformations originating from local writhing (coiling) instability, they will reconnect such that, from elements of tubes, effectively new strings are created, with a new effective entanglement. It is reasonable to assume that these new, episodic, flux tubes, by their mutual linking, satisfy on the average the energy-topology constraints. Therefore we will assume that the field flux tubes inside the stochastic region will reconnect to generate transiently configurations that exhibit a certain topological entanglement. Together with the dynamical nature of the stochasticity phenomena, the formation of these entangled structures is transient and we may suppose that the higher topological content results from a statistical average. At large time the topological reduction occurs with suppression of relative linking via tube merging, a process called by Parker *topological dissipation* [1].

2 Energy and topology of divergenceless vector fields

For two curves γ_1 and γ_2 , the *link* invariant is given by the formula (Gauss)

$$Lk(\gamma_1, \gamma_2) = \frac{1}{4\pi} \oint_{\gamma_1} dx^\mu \oint_{\gamma_2} dy^\nu \varepsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3} \quad (1)$$

This is an integer number and represents the relative entanglement of two magnetic lines in the stochastic region. If a line closes in itself (as a magnetic

line on a resonant surface in tokamak), the formula can still be applied, giving the self-linking. It is obtained by taking $\gamma_1 \equiv \gamma$, $\gamma_2 \equiv \gamma + \varepsilon \hat{\mathbf{n}}$ with $\hat{\mathbf{n}}$ a versor perpendicular to the tangent of γ , and taking the limit $\varepsilon \rightarrow 0$ (this operation is called *framing*). However for a flux tube a more complex situation arises: the magnetic field in the tube has the lines twisted relative to the axis and the topological description is given in terms of the *twist* invariant. It is calculated by considering a line on the surface of the tube and the axis of the tube and defining the vectors: $\mathbf{T}(s)$, the tangent to the axis of the tube; $\mathbf{U}(s)$, the versor from the current point on the axis toward the current point on the line; s is the length along the axis. Then the *twist* is defined as

$$Tw = \oint_{\gamma} ds [\mathbf{T}(s) \times \mathbf{U}(s)] \cdot \frac{d\mathbf{U}(s)}{ds} \quad (2)$$

The deformation of the flux tube of axis γ relative to the plane is measured by the topological number *writhe*, defined as

$$Wr(\gamma) = \frac{1}{4\pi} \oint_{\gamma} dx^{\mu} \oint_{\gamma} dy^{\nu} \varepsilon_{\mu\nu\rho} \frac{(x-y)^{\rho}}{|x-y|^3} \quad (3)$$

While the *twist* measures the rate at which a line twists around the axis of the tube, the *writhe* measures the rate at which the axis of the tube is twisted in space. The following relation exists between the three basic topological numbers for a flux tube

$$Lk = Tw + Wr \quad (4)$$

Instead of generalizing Lk to an arbitrary but finite number of curves (magnetic lines) in space, it is defined an equivalent topological quantity (also noted Lk) which refers this time to a vector field in the volume. Instead of the Gauss link (a discrete set of curves) the definition will now imply a continuous, field-like invariant, the Chern-Simons action, which is the total helicity. For a divergenceless vector field ξ (velocity or magnetic field) in R^3 the helicity is defined as

$$H(\xi) = \int_M d^3x (\xi, \text{curl}^{-1}\xi) \quad (5)$$

which is the same as the integral of $\mathbf{v} \cdot \boldsymbol{\omega}$ or $\mathbf{A} \cdot \mathbf{B}$ over volume. Consider two narrow, linked flux tubes of the vector field ξ , γ_1 and γ_2 . Then

$$H(\xi) = 2Lk(\gamma_1, \gamma_2) \cdot |flux_1| \cdot |flux_2| \quad (6)$$

which shows that the *link* of the magnetic flux tubes is a measure of the magnetic *helicity* in the volume. The total helicity in the volume is the

integral of the Chern-Simons form and with adequate boundary conditions this is an integer number, a direct consequence of the topological nature of the *link* invariant (invariance refers here to deformations of the field $\xi (\equiv \mathbf{B})$ that do not break and reconnect the lines)

$$Q = \frac{1}{32\pi^2} \int d^3x \varepsilon^{jkl} F_{jk} A_l \quad (7)$$

where A_l is the magnetic potential and F_{jk} is the electromagnetic stress tensor. The integrand is the Chern-Simons form, or helicity density, for the magnetic field. The integer Q is called the Hopf invariant. The magnetic lines of a field \mathbf{B} or the streamlines of a flow \mathbf{v} are obtained from equations like $\sum dx_i/B_i = 0$, generally difficult to solve. Therefore using these solutions to construct topological invariants is very difficult and we would need a different representation of the lines for easier handling. This is provided by the Skyrme-Faddeev model, or the modified $O(3)$ nonlinear sigma model, where a line is a topological soliton, clearly exhibiting topological properties (see Ward [2]). It is very suggestive that this model has recently been derived precisely starting from the plasma of electrons and ions, coupled to electromagnetic field (Faddeev and Niemi [3]). It is then legitimate to use the general results derived for the Skyrme-Faddeev model and in particular the following lower bound for the energy of the magnetic field. The inequality is

$$E \geq \eta Q^{3/4} \quad (8)$$

where η is a constant. It means that the energy is bounded from below by the 3/4-th power of the total helicity content in the volume or by a quantity that contains the *total linking* of magnetic lines in the volume, at the power 3/4.

A more practical measure of the topological content is the *average crossing number* C , obtained for a pair of lines by summing the signed intersections in the plane-projection of the spatial curves, averaged over all directions of projection. It differs of Eq.(3) by taking the absolute value of the mixed product. Friedmann and He [4] have extended the concept for a continuous field. We follow the argument of Berger [5] to find the energy bound $E \geq \text{const } C^2$.

Consider magnetic flux tubes whose ends are tied to points situated in two parallel planes (at distance L) and are linked one with the others. Taking two points on their axis, we connect them with a line and measure the angle formed by this line with a fixed direction in one of the plane of projection. This quantity is θ_{12} . The crossing number can be expressed using this angle

$$\bar{c} = \frac{1}{\pi} \int_0^L dz \left| \frac{d\theta_{12}}{dz} \right| \quad (9)$$

The magnetic fields has magnitudes B_{z1} and B_{z2} . We combine energetic and topological quantities by weighting the angle variation along z with the two magnetic fluxes

$$C = \frac{1}{2\pi} \int_0^L dz \int d^2x_1 \int d^2x_2 B_{z1} B_{z2} \left| \frac{d\theta_{12}}{dz} \right| \quad (10)$$

The angle θ_{12} is generated by the deviation of the lines (with tangent versors $\hat{\mathbf{n}}_{1,2}$) relative to a reference straight vertical line normal to the end planes. This deviation is produced by the component of the magnetic field which is perpendicular on the main field B_z , $d\mathbf{x}_1/dz = \mathbf{B}_{\perp 1}(\mathbf{x}_1)/B_0$. Then one finds

$$\frac{d\theta_{12}}{dz} = \frac{1}{r_{12}} (\hat{\mathbf{n}}_2 - \hat{\mathbf{n}}_1) \cdot \hat{\mathbf{e}}_{\theta_{12}} \quad (11)$$

The versor of the line connecting points on the two magnetic field lines is $\hat{\mathbf{e}}_{\theta_{12}} = \hat{\mathbf{e}}_z \times \hat{\mathbf{r}}_{12}$. Then

$$\frac{dC}{dz} = \int \int d^2x_1 d^2x_2 \frac{B_{z1} B_{z2}}{2\pi r_{12}} |(\hat{\mathbf{n}}_2 - \hat{\mathbf{n}}_1) \cdot \hat{\mathbf{e}}_{\theta_{12}}| \quad (12)$$

The energy of the magnetic field in a volume is $E_f = \frac{B_z^2}{2\mu_0} \int d^3x \mathbf{b}^2$ where $\mathbf{b} \equiv (b_x, b_y) = \mathbf{B}_{\perp}/B_z$. By successive bounds Berger finds the inequality

$$E_f \geq \text{const } C^2 \quad (13)$$

We now have two inequalities implying the energy of the magnetic field in a region and two measures of the topological content in that volume: one is the *linking* of the magnetic lines, or equivalently, the helicity H , and the other is a more geometrical characterization of the entanglement, the *average crossing number*, C .

Taking the inequalities as saturated we can estimate the total average crossing number for a certain amount of helicity H in the volume

$$C \sim H^{3/8} \quad (14)$$

The next step is to connect the crossing number with geometrical properties of a generic magnetic field line. For a magnetic field line γ the topological quantity *crossing number* C can be estimated from the number $\Omega(\gamma)$ of intersections of the line γ with an arbitrary surface. There is a general theorem that allows to estimate this number as

$$2\pi\Omega(\gamma) \sim 4K_1 + 3K_2 \quad (15)$$

where $K_{1,2}$ are integrals of Frenet curvatures. Taking $|K|$ as the upper estimation of the local value of the curvature along γ , we have

$$C \sim \Omega(\gamma) \sim L |K| \quad (16)$$

where L is the length of the tube. Then we have

$$|K| \sim H^{3/8} \quad (17)$$

These very qualitative estimations led us to a scaling law connecting the average of the local curvature of a typical magnetic field line with the helicity inside the finite volume of the stochastic region. A magnetic line is curved since it is linked with other lines, and this link is generated for the magnetic structure to be able to store the energy in a stochastic region. Generation of linking also occurs when a certain amount of helicity is injected in a plasma volume. It is reasonable to assume that the curvature is distributed randomly in the volume.

3 Effects of topology on resistivity and diffusion

The curvature of magnetic flux tubes induces drifts of particles. Electrons and ions flowing along curved magnetic lines will have opposite drifts and local charge separations produce random transversal electric fields. For a finite collisionality this is a source of additional dissipation. The equation for ions is

$$ev_{\parallel i} E_z \frac{\partial \tilde{f}_i}{\partial \varepsilon} + ev_{di} \tilde{E}_{\perp} \frac{\partial f_{i0}}{\partial \varepsilon} = -\nu \tilde{f}_i \quad (18)$$

for a transversal field $\tilde{E}_{\perp} \sim \eta_0 n_0 \frac{T_e}{B} (1 + \tau) |K|$ and $|K| = |(\hat{\mathbf{n}} \cdot \nabla) \hat{\mathbf{n}}|$, for which we can use an estimation based on Eq.(17). We finally obtain an estimation of the negative current perturbation due to curvature

$$|\delta \tilde{j}_{\parallel}| \sim e \frac{\eta_0}{\nu} \frac{n_0^2 T_e}{B^2} \sqrt{\frac{T_i}{m_i}} (1 + \tau) |K|^2 \frac{1 + \rho + \rho^2}{1 + \rho} \quad (19)$$

with $\rho \equiv \left(\frac{e_i v_{\parallel i} E_z}{\nu T_i} \right)^{1/2}$. This is not a substantial modification of the equilibrium current (less than 1%), which means that the enhanced resistivity is mainly due to other processes.

In general it is assumed that the magnetic reconnection does not affect the total helicity. However there is a dynamic redistribution of helicity (with

overall conservation) since in the stochastic region there are filaments of current and local increase of the parallel electric field, from which we have $dh/dt = -2\mathbf{E} \cdot \mathbf{B}$. Then this mechanism is a potential feedback loop: higher resistivity leads to higher reconnection rate and higher helicity perturbation, which in turn creates magnetic linking and curvature.

In an alternative approach to the problem of topology of the magnetic field in a stochastic region, we can base our estimations on the much simpler assumption, that a magnetic line is randomly “scattered” at equal space (z) intervals and performs a random walk. Actually this is the classical assumption for the diffusion in stochastic magnetic fields. To characterize quantitatively the topology of the line we use the analogy with the polymer entanglement. A functional integral formalism can be developed (Tanaka [6]) taking as action the free random walk, with the constraint that the line has a fixed, m , winding number around a reference straight line. The mean square dispersion of the winding number (linking) can then be calculated

$$\langle m^2 \rangle \sim \frac{1}{4\pi} (\ln N)^2 \quad (20)$$

where N is the number of steps. Since k_{\parallel} is low for magnetic turbulence, the winding number is a small number. But this represents the random winding naturally occurring in an unconstrained random walk of the magnetic line, when the magnetic perturbation is a Gaussian noise. Actually, we know that a given amount of helicity can only be realised by a certain volume-averaged mutual linking and this is an effective constraint which can only be realised through a much higher density of $\sqrt{\langle m^2 \rangle}$ than the free random motion. Then the higher winding leads to a sort of trapping for the magnetic lines and the effective diffusion will be smaller than for the brownian case [7].

In general the use of topological quantities can improve the description of stochastic magnetic fields, *e.g.* diffusion, Kolmogorov length, etc. These are usually expressed in terms of mean square amplitude of the perturbation, $\langle |\tilde{b}|^2 \rangle$ but including the topological quantities can lead to more refined models.

Consider the line γ of a perturbed magnetic field and the equation $Du = 0$ where D is the covariant derivative, similar to the velocity, $v = p - A$ which is applied on a function u along the line γ . The equation says that the covariant derivative along the line, of the function u is zero. Then $(\frac{\partial}{\partial s} - iA)u = 0$ leads to $u = u_0 \exp\left(i \oint_{\gamma} ds A(s)\right)$. It is natural to make a generalization of the two-dimensional concept of point-like vortices and introduce the spinors

along the magnetic line γ . By the same arguments (Spineanu and Vlad [8]) we will need the dual (dotted-indices) spinors and we need to represent A in $SU(2)$. Then, more generally, the expression of u is

$$W_\gamma \equiv \text{Tr}_R P \exp \left(i \oint_\gamma dx^\mu A_\mu \right) \quad (21)$$

the trace is over the representation R of $SU(2)$ and P is the ordering along γ . Being a closed path this number is a functional of γ and of A . This is the Wilson loop.

We subject the fluctuations of the potential A to the constraint of minimum helicity in the volume, because lower helicity allows lower energy according to the bound Eq.(8). The Boltzmann weight in the partition function is then the exponential of an *action* representing the total helicity, *i.e.* the integral of the Chern-Simons density (compared with Eq.(7), here A is a matrix)

$$S = \frac{\kappa}{4\pi} \int_{M^3} d^3r \varepsilon^{\mu\nu\rho} \text{Tr} \left(A_\mu \partial_\nu A_\rho - \frac{2}{3} A_\mu A_\nu A_\rho \right) \quad (22)$$

The average of W_γ is

$$\int D[A] W_\gamma \exp(S) \quad (23)$$

can be expanded in powers κ^{-n} . The first significant term (order κ^{-1}) is the integral of the two-point correlation of the fluctuating potential

$$- \text{Tr} \left[(R^a R^b) \oint_\gamma dx^\mu \int^x dy^\nu \langle A_\nu^a(y) A_\mu^b(x) \rangle \right] \sim Wr(\gamma) \quad (24)$$

where $Wr(\gamma)$ is the *writhe* number of the curve γ (a, b are labels in $SU(2)$). Calculating Lk =self-linking of the curve γ , the classical relation is obtained between the *link*, the *twist* and the *writhe* $Lk(\gamma) = Tw(\gamma) + Wr(\gamma)$.

Therefore if the fluctuation of the poloidal flux function ψ , or the z -component of the magnetic potential A_z are such that higher helicity states are difficult to access, then the two-point correlations of the perturbed potential along a curve γ can be expressed as the kernel of the Gauss integral for the self-linking number of γ . The result

$$\langle \psi(x_1) \psi(x_2) \rangle_\gamma \sim \text{integrand of } Lk(\gamma) \quad (25)$$

possibly sheds a new light on the correlations of fluctuating magnetic quantities, since we now express them also by topological quantities.

4 Discussion

We have examined the topological constraints on the stochastic magnetic configuration when a transient increase of helicity occurs in a finite plasma volume. Via bounds related to the magnetic energy that can be safely stored in that volume (*i.e.* a statistical stationarity can be attained) a scaling can be derived between the helicity and the average curvature of a generic magnetic line in the volume. The particles' curvature-drift-induced new dissipation appears to not modify substantially the resistivity. However the new instruments that imply the topology of magnetic field are useful : the average dispersion of the winding of a line relative to a reference axis serves to quantify the trapping of a line and the reduction of the classical magnetic diffusion.

We note finally that using these analytical instruments the *topological dissipation* process may be described by the coupling of the magnetic helicity density (the Chern-Simons Lagrangian density) with a pseudoscalar field. The dynamics of this field is that of the kinematic helicity of the plasma and again a field-theoretical description appears to be possible.

References

- [1] PARKER, E.N., Astrophys. J. **174** (1972) 499.
- [2] WARD, R. S. , Nonlinearity 12 (1999) 241.
- [3] FADDEEV, L., A.J. NIEMI, Nature 387 (1997) 58.
- [4] FRIEDMANN, M. H., Z.-X. HE, Ann. Math. 134 (1991) 189.
- [5] BERGER, M. A., Astron. Astrophys. 252 (1991) 369.
- [6] TANAKA, F., Prog. Theor. Phys. 68 (1982) 148.
- [7] SPINEANU, F., M. VLAD, Phys. Plasmas 4 (1997) 2106.
- [8] SPINEANU, F. , M. VLAD, Phys. Rev. Lett. 94 (2005) 235003.